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Chiral Anomalies in the Stochastic Quantization Scheme★

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Abstract. Stochastic quantization scheme with an invariant stochastic regularization is applied to models of chiral fermions in even-dimensional spacetime. In spite of the manifest preservation of chiral gauge symmetries by stochastic regularization of the stochastic averages, the anomalous chiral Ward identities are correctly reproduced in a covariant form after removing the regularization.

1. In the present Letter we consider the derivation of chiral anomalies (for a recent account, see [1]) in arbitrary even (Euclidean) spacetime dimensions D within a stochastic quantization scheme (SQS) [2]. Here, the SQS is carried over to the case of chiral fermions. One of the most intriguing properties of an SQS is the possibility of introducing a new nonperturbative regularization affecting only the stochastic time direction [3] and, thus, implying that all symmetries of the models, including gauge-, chiral-, and supersymmetries, are preserved in the stochastic averages. As pointed out in [3], an SQS introduces new nonlocal terms which, although preserving all the symmetries in the stochastic averages, spoil the standard form of the corresponding Ward identities (see Equation (13) below). It is, therefore, a nontrivial task to check whether or not the correct Ward identities are reproduced in the equilibrium limit of the SQS with stochastic regularization.

The regularized anomalous divergence of induced chiral fermion currents within the SQS is expressed through integrals over the stochastic time of the heat kernels of $\mathscr{D}\mathscr{D}^*$ and $\mathscr{D}^*\mathscr{D}$, where $\mathscr{D}^* \equiv \mathscr{D}^*(A)$ is the left-handed chiral Dirac operator in a background gauge field $A_\mu(x)$. The final removal of the SQS regularization reduces the problem to an analysis of the small-time behaviour of the heat kernels. Thus, the present SQS approach resembles the heat kernel derivation of anomalies in [4]. Although SQS regularization manifestly preserves the chiral gauge symmetries in the stochastic averages, the anomalous chiral Ward identities are correctly reproduced (after removing the regularization) in the covariant (not consistent with the Wess–Zumino integrability

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conditions [5], see also [6]) form. In Section 3 below, it is explained why these latter properties of the SQS do *not* contradict each other.

2. In this section we present a brief formulation of the SQS for chiral (e.g., left-handed) fermion fields $\psi_L(\tau, x)$ (for Dirac Fermions ψ, see [3, 7]), belonging to the fundamental representation of U(n) and interacting with a background U(n) gauge field $A_\mu(x)$ (D = even). We take the Langevin equations (the basic ingredients of SQS [2]) in the form

$$\partial_\tau \psi_L^\perp = -(\mathscr{D}\mathscr{D}^*)\psi_L^\perp + \eta_L^\perp\,, \qquad \partial_\tau \bar{\psi}_L^\perp = -(\mathscr{D}^*\mathscr{D})^T \bar{\psi}_L^\perp + \bar{\eta}_L^\perp\,; \tag{1}$$

$$\langle \eta_L^\perp(\tau, x)\bar{\eta}_L^\perp(\tau', x')\rangle = -2i\,\delta_\Lambda(\tau - \tau')\mathscr{D}(1 - \Pi_0)\,\delta^{(D)}(x - x')\,. \tag{2}$$

Here and below, the following notations are used

$$\mathscr{D}^{(*)} \equiv \mathscr{D}^{(*)}(A) = i\sigma_\mu^{(*)}\nabla_\mu(A) = i\sigma_\mu^{(*)}(\partial_\mu + iA_\mu(x))\,,$$
$$\sigma_\mu\sigma_\nu^* + \sigma_\nu\sigma_\mu^* = \sigma_\mu^*\sigma_\nu + \sigma_\nu^*\sigma_\mu = \delta_{\mu\nu} \quad (\sigma_\mu^{(*)} - 2^{(D/2)-1} \times 2^{(D/2)-1}\ \text{matrices})\,; \tag{3}$$

$$A_\mu(x) = A_\mu^a(x)T^a\,, \qquad \mathrm{tr}(T^aT^b) = n\delta^{ab}\,,$$
$$T^aT^b = \delta^{ab} + (d^{abc} + if^{abc})T^c\,, \tag{4}$$
$$x \in \mathbb{R}^D\,, \qquad \mu, \nu = 0, 1, \ldots, D - 1\,, \qquad a, b, c = 0, 1, \ldots, n^2 - 1\,;$$

$$\psi_L^\perp \equiv (1 - \bar{\Pi}_0)\psi_L\,, \qquad \bar{\psi}_L^\perp \equiv (1 - \Pi_0)\bar{\psi}_L \quad \text{and analogously for } \bar{\eta}_L^\perp\,. \tag{5}$$

T^a in (4) denotes the Hermitian U(n) generators, with T^0 belonging to the U(1) subalgebra★. The superscript 'T' in (1) means operator transposition. Summation over repeated indices is understood and the latter will be suppressed in what follows. The Π_0, $\bar{\Pi}_0$ in (2) and (5) are the zero-mode projectors of $\mathscr{D}^*\mathscr{D}$ and $\mathscr{D}\mathscr{D}^*$, respectively. Notations (3) assume the following representation of Euclidean Dirac matrices and of the full Dirac operator $\nabla(A)$:

$$\gamma_\mu = i\begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu^* & 0 \end{pmatrix}, \qquad \gamma^{(D+1)} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$
$$\nabla(A) = \gamma_\mu\nabla_\mu(A) = \begin{pmatrix} 0 & \mathscr{D} \\ \mathscr{D}^* & 0 \end{pmatrix}, \qquad \psi_L = \tfrac{1}{2}(\mathbf{1} + \gamma^{(D+1)})\psi\,. \tag{6}$$

In (1), the anticommuting chiral spinor field, η_L denotes a Gaussian random source with the two-point correlation function given by (2).

The SQS regularization proposed in [3] consists of inserting into (2) a regularized δ-function $\delta_\Lambda(\tau - \tau')$ obeying the following properties

$$\lim_{\Lambda \to \infty} \delta_\Lambda(\tau) = \delta(\tau)\,, \qquad \delta_\Lambda(-\tau) = \delta_\Lambda(\tau)\,,$$
$$\left.\frac{\mathrm{d}^k}{\mathrm{d}\tau^k}\,\delta_\Lambda(\tau)\right|_{\tau=0} = 0\,, \quad k = 0, 1, \ldots, L - 1\,, \tag{7}$$

where L is an appropriate integer.

★ f^{abc} and d^{abc} in (4) vanish identically if any of the indices takes a zero value.

The reason for projecting out the SQS equations (1) and (2) by (5) is to guarantee the approach to equilibrium ($\mathscr{D}\mathscr{D}^*$ and $\mathscr{D}^*\mathscr{D}$ in (1) should be positive in order to yield a 'drift force' [2])

$$\langle F[\bar{\psi}_L^{\perp}(x)]\rangle_Q = \lim_{\tau\to\infty} \langle F[\bar{\psi}_L^{\perp}(\tau, x)]\rangle_\eta . \tag{8}$$

The subscript 'Q' on the left-hand side of (8) denotes the usual Euclidean quantum average of an arbitrary functional $F[\bar{\psi}_L^{\perp}(x)]$ with the regularized weight

$$\exp\{-S_\Lambda\} = \exp\left\{-i\int \mathrm{d}^D x\, \mathrm{d}^D x'\, \bar{\psi}_L^{\perp}(x)\mathscr{D}_\Lambda^*(x, x')\psi_L^{\perp}(x')\right\},$$

$$\mathscr{D}_\Lambda^{*-1}(x, x') = \mathscr{D}(\mathscr{D}^*\mathscr{D})^{-1}\left\{2\int_0^\infty \mathrm{d}\tau\, \delta_\Lambda(\tau)\exp[-\tau\mathscr{D}^*\mathscr{D}](1-\Pi_0)\right\}(x, x'), \tag{9}$$

$$\mathscr{D}_\Lambda^* \xrightarrow[\Lambda\to\infty]{} \mathscr{D}^* .$$

The subscript 'η' on the right-hand side of (8) indicates the SQS average according to (2) with $\bar{\psi}_L^{\perp}(\tau, x)$ being the solution of (1) subject to arbitrary initial conditions $\bar{\psi}_L^{\mathrm{in}}(\tau_0, x)$. Choosing $\bar{\psi}_L^{\perp\,\mathrm{in}} = 0$ at $\tau_0 = -\infty$ we have

$$\begin{aligned}
&\psi_L^{\perp}(\tau, x) = (G_L\eta_L^{\perp})(\tau, x), \qquad \bar{\psi}_L^{\perp}(\tau, x) = (\bar{G}_L^T\bar{\eta}_L^{\perp})(\tau, x),\\
&G_L(\tau, x; \tau', x') = \theta(\tau-\tau')\exp\{-(\tau-\tau')\mathscr{D}\mathscr{D}^*\}(x, x'),\\
&\bar{G}_L(\tau, x; \tau', x') = \theta(\tau-\tau')\exp\{-(\tau-\tau')\mathscr{D}^*\mathscr{D}\}(x, x').
\end{aligned} \tag{10}$$

In case of (10), the limit ($\tau\to\infty$) in (8) turns out to be irrelevant.

Let us emphasize that the invariance under the chiral gauge transformation

$$\begin{aligned}
&\psi_L \xrightarrow{g} \psi_L(\tau, x\,|\,A_\mu^g, \eta_L^g) = g^{-1}(x)\psi_L(\tau, x\,|\,A_\mu, \eta_L),\\
&\bar{\psi}_L \xrightarrow{g} \bar{\psi}_L(\tau, x\,|\,A_\mu^g, \bar{\eta}_L^g) = \bar{\psi}_L(\tau, x\,|\,A_\mu, \bar{\eta}_L)g(x),\\
&\mathscr{D}^{(*)}(A^g) = g^{-1}\mathscr{D}^{(*)}(A)g, \qquad g(x)\in \mathrm{U}(n), \quad g(x)\to \mathbf{1}, \ |x|\to\infty,\\
&A_\mu^g(x) = g^{-1}(x)(A_\mu(x) - i\,\partial_\mu)g(x), \qquad \eta_L^g(\tau, x) = g^{-1}(x)\eta_L(\tau, x),\\
&\bar{\eta}_L^g(\tau, x) = \bar{\eta}_L(\tau, x)g(x),
\end{aligned}$$

is manifestly preserved in (1), (10) (2), regularized by (7) (here the functional dependence of the solutions of (1) $\bar{\psi}_L^{\perp}(\tau, x)$ on $A_\mu(x)$, $\bar{\eta}_L(\tau, x)$ is explicitly indicated).

To be precise, we shall assume standard boundary conditions for $A_\mu(x)$ (allowing compactification of the problem from $\mathbb{R}^D$ to S^D)

$$A_\mu(x) = -ih^{-1}(\hat{x})(\partial_\mu h)(\hat{x}) + O(|x|^{-1-\varepsilon}) \quad \text{for } |x|\to\infty, \tag{11}$$

$$\hat{x} = \frac{x}{|x|}\in S^{D-1}, \qquad h: S^{D-1}\to \mathrm{U}(n).$$

Also, the following identities will be useful for calculating the SQS averages (8) in the next section

$$\overline{\Pi}_0 \mathscr{D} = \mathscr{D}\Pi_0 , \qquad \mathscr{D}^*\overline{\Pi}_0 = \Pi_0 \mathscr{D}^* ,$$
$$\mathscr{D} \exp[-\tau\mathscr{D}^*\mathscr{D}] = \exp[-\tau\mathscr{D}\mathscr{D}^*]\mathscr{D} , \quad \mathscr{D}^* \exp[-\tau\mathscr{D}\mathscr{D}^*] = \exp[-\tau\mathscr{D}^*\mathscr{D}]\mathscr{D}^* . \tag{12}$$

3. Let us consider the SQS regularized covariant divergence of the induced chiral fermion current

$$\begin{aligned}\nabla_\mu^{ab} J_\mu^{L,b}(x) &\equiv \nabla_\mu^{ab} \langle \overline{\psi}_L^\perp(\tau, x)\sigma_\mu^{(*)}\psi_L^\perp(\tau, x)\rangle_\eta \\ &= i \langle\{[\mathscr{D}^*(\mathscr{D}\mathscr{D}^*)^{-1}]^T(\overline{\eta}_L^\perp - \partial_\tau\overline{\psi}_L^\perp)\}(\tau, x)\psi_L^\perp(\tau, x)\rangle_\eta - \\ &\quad - i \langle\overline{\psi}_L^\perp(\tau, x)\{\mathscr{D}^*(\mathscr{D}\mathscr{D}^*)^{-1}](\eta_L^\perp - \partial_\tau\psi_L^\perp)\}(\tau, x)\rangle_\eta ,\end{aligned} \tag{13}$$

where $\nabla_\mu^{ab} = \delta^{ab}\partial_\mu + f^{abc}A_\mu^c(x)$ and Equations (1) were used. Substituting (10) into the right-hand side of (13) and performing a random source average according to (2), we arrive at the following expression (which does not already depend explicitly on the stochastic time)

$$\begin{aligned}\nabla_\mu^{ab} J_\mu^{L,b}(x) = 2 \int_0^\infty & \mathrm{d}\tau\, \delta_\Lambda(\tau)\, \mathrm{tr}\{T^a[\exp[-\tau\mathscr{D}^*\mathscr{D}](1 - \Pi_0)(x, x) - \\ & - \exp[-\tau\mathscr{D}\mathscr{D}^*](1 - \overline{\Pi}_0)(x, x)]\} .\end{aligned} \tag{14}$$

To obtain Equation (14), identities (12) as well as the identity true for any nonnegative elliptic (selfadjoint) operator H

$$\int_{\tau_1}^{\tau_2} \mathrm{d}\tau \exp[-\tau H] = H^{-1}(\exp[-\tau_1 H] - \exp[-\tau_2 H])$$

were employed.

The presence of $\delta_\Lambda(\tau)$ (7) in (14) regulates all eventual ultraviolet divergences, i.e., singularities of the form $O(\tau^{-k})$, $k \geqslant 1$, at the lower limit of the 'proper-time' integral. One of the most efficient tools to analyze the latter is the well-known asymptotic Seeley expansion [8] for the heat kernel

$$\exp[-\tau H](x, x) = \sum_{j=0}^{\infty} \tau^{1/r(j-D)}\, \Phi_{1/r(j-D)}^{(D)}(H; x) , \tag{15}$$

where r is the order of H and $\Phi_{1/r(j-D)}^{(D)}(H; x)$ are local functionals of the coefficients of H. Some relevant properties of (15) for $H = \nabla^2(A)$ are briefly recalled in the Appendix.

Using (A1)–(A5) it may be easily shown that (accounting for (6))

$$\begin{aligned}&\mathrm{tr}[T^a(\Phi_{1/2(j-D)}^{(D)}(\mathscr{D}\mathscr{D}^*; x) - \Phi_{1/2(j-D)}^{(D)}(\mathscr{D}^*\mathscr{D}; x))] \\ &\quad = \mathrm{tr}[T^a\gamma^{(D+1)}\Phi_{1/2(j-D)}^{(D)}(\nabla^2(A); x)] = 0 , \quad \text{for } j < D .\end{aligned}$$

Therefore, no ultraviolet divergences arise in (14) after taking the limit $\Lambda \to \infty$★ (cf. (7))

$$\begin{aligned}\nabla_\mu^{ab} J_\mu^{L,b}(x) &= \mathrm{tr}[T^a \gamma^{(D+1)} \Pi_0^{\nabla(A)}(x,x)] - \\ &\quad + \mathrm{tr}[T^a(\Phi_0^{(D)}(\mathscr{D}^*\mathscr{D};x) - \Phi_0^{(D)}(\mathscr{D}\mathscr{D}^*;x))] \\ &= \mathrm{tr}[T^a\gamma^{(D+1)}\Pi_0^{\nabla(A)}(x,x)] - \mathrm{tr}[T^a\gamma^{(D+1)}\Phi_0^{(D)}(\nabla^2(A);x)]\,, \end{aligned} \tag{16}$$

where in the last line once again (6) was accounted for and $\Pi_0^{\nabla(A)}$ denotes the zero-mode projector of the full Dirac operator.

By a straightforward computation (using (A1)–(A5)) of the last term in (16), we finally get the covariant form [9, 1] of the U(n) non-Abelian chiral anomaly

$$\begin{aligned}\nabla_\mu^{ab} J_\mu^{L,b}(x) &= \mathrm{tr}[T^a \gamma^{(D+1)} \Pi_0^{\nabla(A)}(x,x)] - \\ &\quad - \left[\left(\frac{D}{2}!\,(4\pi)^{D/2}\right]^{-1} \varepsilon_{\mu_1 \ldots \mu_D}\, \mathrm{tr}[T^a F_{\mu_1\mu_2}\ldots F_{\mu_{D-1}\mu_D}]\,; \end{aligned} \tag{17}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]\,.$$

In particular, for the U(1) subgroup (17) reads (e.g., [4, 1]):

$$\partial_\mu J_\mu^{L,a=0}(x) = \mathrm{index}(\mathscr{D}^*;x) - C_{D/2}(F;x)\,, \tag{18}$$

where index $(\mathscr{D}^*;x)$ denotes the index density of $\mathscr{D}^*$ and $C_{D/2}(F;x)$ is the density of the $D/2$th Chern characteristic class (see, e.g., [10]).

Thus, we have shown that the SQS for chiral fermions yields the correct covariant form of the (non-Abelian) chiral anomalies in spite of the fact that the intermediate SQS regularization [3] manifestly respects the chiral gauge symmetries in the stochastic averages (8). To understand why these two properties of SQS are compatible, let us recall that the (regularized) anomalous chiral Ward identities result from the chiral gauge noninvariance of the regularized effective fermion action (for a discussion, see the second ref. [12]) which in the present case of stochastic regularization reads (cf. (9))

$$S_\Lambda^{\mathrm{eff}}[A] = -\ln\det[-i\mathscr{D}_\Lambda^*(A)]\,, \qquad S_\Lambda^{\mathrm{eff}}[A^g] \neq S_\Lambda^{\mathrm{eff}}[A]\,,$$

and hence

$$\left.\frac{\delta S_\Lambda^{\mathrm{eff}}[A^g]}{\delta g(x)}\right|_{g(x)=1} = i\nabla_\mu J^L_{\mu,\Lambda}(x) \neq 0\,.{}^{\star\star} \tag{19}$$

On the other hand, however, let us emphasize that there does *not* exist a functional $F(\bar\psi_L^\perp(\tau,x))$ such that $S_\Lambda^{\mathrm{eff}}[A]$ (19) could be obtained as an equilibrium limit of an SQS average (8). Therefore, the chiral gauge noninvariance of $S_\Lambda^{\mathrm{eff}}[A]$ does not contradict the manifest chiral gauge invariance of (8).

★ The regularizing $\delta_\Lambda(\tau)$ plays a crucial role in the course of the derivation of (14). Just because of its presence, both operations (a) taking the matrix trace and (b) taking integrals over the stochastic time, were mathematically correctly interchanged.

★★ Here $J^L_{\mu,\Lambda}$ denotes the consistent current which differs from the covariant one J_μ^L (13) by a local finite counterterm [6].

Finally, let us note that the SQS exhibits a serious drawback in odd D. As shown in [11], the SQS fails to reproduce the existing parity-violating anomalies [12] of the odd-dimensional massless fermions.

Due to technical reasons, the submission of this Letter (containing part of the material of the preprint [11]) was delayed by several months. Meanwhile, [13] appeared which discusses similar problems to the ones above. However, these authors' conclusion about noncommutativity of the equilibrium limit (8) and the massless limit for Dirac fermions, leading to a cancellation of the chiral anomalies within an SQS for finite stochastic time, is wrong. This is due to an elementary mathematical error in Equation (13) of ref. [13] (for a criticism, see [14]).

Appendix

Here we shall list some useful formulae concerning the computation of the coefficients $\Phi^{(D)}_{1/2(j-D)}(\nabla^2(A); x)$ in the Seeley expansion (15) for the heat kernel of $\nabla^2(A)$ where the standard boundary conditions (11) are assumed for $A_\mu(x)$.

Let us introduce the symbols $\sigma(x; \xi, \lambda)$, $R(x; \xi, \lambda)$ of $\nabla^2(A) - \lambda$ and its inverse (parametrix):

$$[\nabla^2(A) - \lambda]\,\delta^{(D)}(x - x') = (2\pi)^{-D} \int \mathrm{d}^D \xi \exp[i\xi(\mathrm{x} - \mathrm{x}')]\,\sigma(\mathrm{x}; \xi, \lambda)$$

and analogously for $R(x; \xi, \lambda)$ (for general notions and proofs in the symbol calculus of elliptic operators, see e.g., [15]). Then, the following (asymptotic) expansions are valid

$$\begin{aligned}
&\sigma(x; \xi, \lambda) = \sum_{k=0}^{2} \sigma_k(x; \xi, \lambda), \qquad \sigma_k(x; \rho\xi, \rho^2\lambda) = \rho^k \sigma_k(x; \xi, \lambda) \quad (\rho > 0),\\
&\sigma_2(x; \xi, \lambda) = \xi^2 - \lambda, \qquad \sigma_1(x; \xi, \lambda) = 2\xi_\mu A_\mu(x),\\
&\sigma_0(x; \xi, \lambda) = -i(\partial_\mu A_\mu) + A_\mu A_\mu + \frac{i}{4}[\gamma_\mu, \gamma_\nu]F_{\mu\nu};\\
&R(x; \xi, \lambda) = \sum_{j=0}^{\infty} R_{-2-j}(x; \xi, \lambda), \quad R_{-2-j}(x; \rho\xi, \rho^2\lambda) = \rho^{-2-j} R_{-2-j}(x; \xi, \lambda),
\end{aligned} \tag{A1}$$

and $R_{-2-j}(x; \xi, \lambda)$ are recursively determined from

$$\delta_{0l} = \sum_{j+k+|\alpha|=l} (\alpha!)^{-1} (\partial_\xi^\alpha \sigma_{2-k}) [(-i\,\partial_x)^\alpha R_{-2-j}], \quad l = 0, 1, 2, \ldots, \tag{A2}$$

where α is a multiindex. Finally, the explicit expressions of $\Phi^{(D)}_{1/2(j-D)}(\nabla^2(A); x)$ are given by

$$\Phi^{(D)}_{1/2(j-D)}(\nabla^2(A); x) = i(2\pi)^{-(D+1)} \int \mathrm{d}^D \xi \int_\Gamma \mathrm{d}\lambda\, \mathrm{e}^{-\lambda} R_{-2-j}(x; \xi, \lambda) \tag{A3}$$

with a contour Γ in the complex λ-plane defined as

$$\Gamma = (i\infty, i\varepsilon) \cup C_\varepsilon \cup (-i\varepsilon, -i\infty).$$

Here C_ε denotes a half-circle of the radius ε connecting the points $i\varepsilon$ and $-i\varepsilon$ and passing through the point $-\varepsilon$.

From (A1) and (A2) one easily finds

$$R_{-2-2l}(x; \xi, \lambda) = (-1)^l(\xi^2 - \lambda)^{-(l+1)}\left(\frac{i}{4}[\gamma_\mu, \gamma_\nu]F_{\mu\nu}\right)^l + \cdots,$$

$$\begin{aligned} &R_{-2-(2l+1)}(x; \xi, \lambda) \\ &\quad = 2i(-1)^l(\xi^2 - \lambda)^{-(l+2)}\sum_{r=0}^{l}\left(\frac{i}{4}[\gamma_\mu, \gamma_\nu]F_{\mu\nu}\right)^r \times \\ &\qquad \times \xi_\lambda(\partial_\lambda + iA_\mu(x))\left(\frac{i}{4}[\gamma_\mu, \gamma_\nu)F_{\mu\nu}\right)^{l-r} + \cdots, \end{aligned} \tag{A4}$$

where the dots stand for terms containing less than $2l$ Dirac matrices.

The following identities are also used in the text (D = even):

$$\begin{aligned} &\mathrm{tr}(\gamma^{(D+1)}\gamma_{\mu_1}\cdots\gamma_{\mu_k}) = 0 \quad (k < D), \\ &\mathrm{tr}(\gamma^{(D+1)}\gamma_{\mu_1}\cdots\gamma_{\mu_D}) = 2^{D/2}(-i)^{1/2D(D+1)}\varepsilon_{\mu_1\ldots\mu_D}. \end{aligned} \tag{A5}$$

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